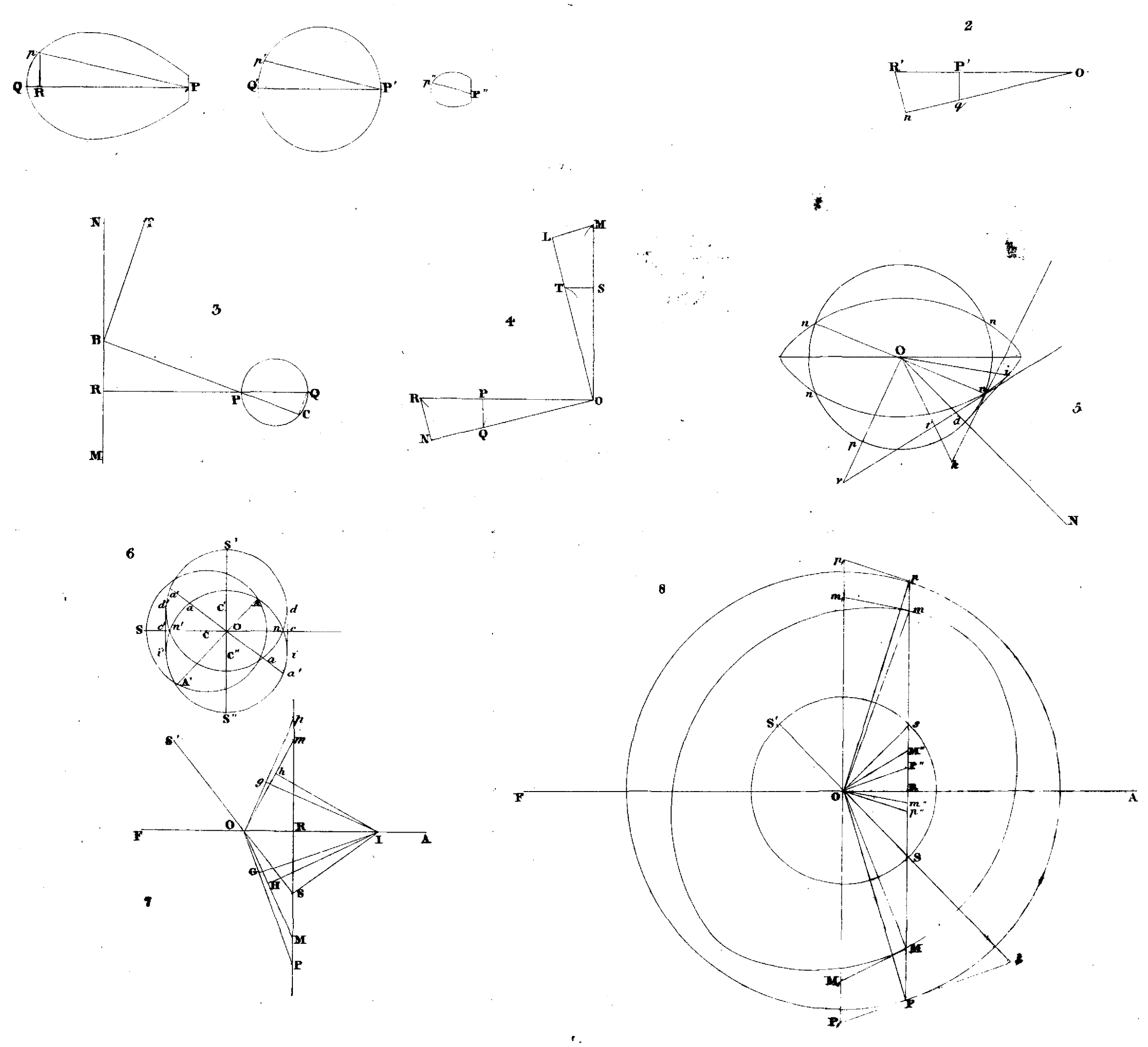
Plate

Fig 1.



Geometrical Propositions applied to the Wave Theory of Light.
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PART I.—GEOMETRICAL PROPOSITIONS.

1. THEOREM I. Conceive a curved surface B to be generated from a given curved surface A in the following manner: having assumed a fixed origin O, apply a tangent plane at any point Q of the given surface, and perpendicular to this plane draw a right line OPR cutting the plane in P, and terminated in R, so that OP and OR may be reciprocally proportional to each other, their rectangle being equal to a constant quantity k^2, and let all the points R taken according to this law generate the second surface B. *Then the relation between these two surfaces, and between the points Q and R, will be reciprocral;* that is to say, if a tangent plane be applied at the point R of the second surface, a perpendicular ON to this plane will pass through the point Q of the first surface, and ON and OQ will be reciprocally proportional to each other, the rectangle under them being also equal to k^2.

2. To prove this theorem, take a point q, in the tangent plane of the surface A, and near the point of contact Q. (*Fig.* 2.) Through q let several other planes be drawn touching the surface A in points Q', Q'', Q''', &c. and draw the perpendiculars $OP'R'$, $OP''R''$, $OP'''R'''$, &c. according to the same law as OPR. The points R, R', R'', R''', &c., will thus be upon the second surface B, and they will moreover be all in the same plane; for from any one of them R' let $R'n$ be drawn perpendicular to the right line Oq and meeting Oq in n; then on account of the similar right-angled triangles $OP'q$ and OnR', the rectangle nOq will be equal to the rectangle $R'OP'$, or to the constant quantity k^2, so that the point n, or the foot of the perpendicular let fall upon Oq, will be the same for all the points R, R', R'', R''', &c., and consequently all these points will lie in a plane cutting the right line Oqn perpendicularly in n, so as to make the rectangle nOq equal to k^2. Now while the point Q remains fixed, let the point q approach to it without limit in the tangent

plane at Q; and the points R', R'', R''', &c. will in like manner approach without limit to the fixed point R; the plane which contains all those neighbouring points having for its limiting position the tangent plane at R. Also the point n will ultimately coincide with N. It follows therefore that the tangent plane at R cuts the right line OQ perpendicularly in N, so as to make the rectangle NOQ equal to k^2.

3. *Corollary.* If any point Q upon the surface A should be a point of intersection, where the surface admits an infinite number of tangent planes, the perpendiculars from O upon these planes will form a conical surface having O for its vertex. In OQ take, as before, a point N, so that $ON \times OQ = k^2$, and let a plane passing through N at right angles to OQ cut the conical surface. The intersection will be a certain curve. From the preceding demonstration it is evident that every point of this curve belongs to the surface B, and that the plane which touches this surface at any point of the curve cuts OQ perpendicularly in N; or, in other words, that *the same plane touches the surface* B *through the whole extent of the curve.*

4. Two surfaces related to each other like A and B in the preceding theorem may be called *reciprocal surfaces*, and points like Q and R *reciprocal points*; the *radii* OQ and OR may likewise be termed reciprocal. A familiar example of such surfaces is afforded, as I have shown on a former occasion*, by two ellipsoids having a common centre at the point O, and their semi-axes coincident in direction, and connected by the relation $aa' = bb' = cc' = k^2$; where a, b, c, are the semi-axes of one ellipsoid in the order of their magnitude, a being the greatest; and a', b', c', those of the other ellipsoid, a' being the least. The mean semi-axes b and b' coincide, and the circular sections of both ellipsoids pass through the common direction of b and b'.

5. It has also been shown with regard to those ellipsoids, that if Q and R be reciprocal points on the surfaces of abc and $a'b'c'$ respectively, and if a right line Oqr, perpendicular to the plane QOR, cut the first ellipsoid in q and the second in r, the lines OQ and Oq will be the semi-axes of the section made in the ellipsoid abc by a plane passing through them; and the lines OR and Or, in like manner, will be the semi-axes of the section made in the other ellipsoid $a'b'c'$ by the plane in which they lie.

6. It may further be remarked, that if the radius OQ in one of the reciprocal ellipsoids describe a plane, the corresponding radius OR will describe another plane. For the planes touching the ellipsoid abc in the points Q will all be parallel to a certain right line, and therefore the perpendiculars OR to these tangent planes will all lie in a plane perpendicular to that right line. These two planes, containing the reciprocal radii, may, for brevity, be called *reciprocal planes*.

When two reciprocal radii lie in a principal plane, at right angles to a semi-axis of

* Transactions of the Royal Irish Academy, Vol. XVI. Part II. pp. 67, 68.

the ellipsoids, it is evident that two planes intersecting in this semi-axis and passing through the reciprocal radii, are reciprocal planes.

7. Theorem II. If three right lines at right angles to each other pass through a fixed point O, so that two of them are confined to given planes; the plane of these two, in all its positions, touches the surface of a cone whose sections, parallel to the given planes, are parabolas; while the third right line describes another cone, whose sections parallel to the given planes are circles.

Let the plane of the figure, (*Fig.* 3) supposed parallel to one of the given planes, be intersected by the other given plane in the right line MN; and let OQ be perpendicular to the latter plane, while OP is perpendicular to the former and to the plane of the figure, so that PQ being joined will meet MN at right angles in R. Let OA, OB, OC, be the three perpendicular lines, of which OA is parallel to the plane of the figure; this plane will be intersected by the plane of OA and OB in a right line BT parallel to OA, and therefore perpendicular to both OB and OP, and to the plane BOP, and to the line BP. Thus the angle PBT is always a right angle, and therefore BT always touches the parabola whose focus is P and vertex R; or, which comes to the same thing, the plane $AOBT$ always touches the cone which has O for its vertex, and the parabola for its section.

Again, since OB, OP, OC, are all at right angles to OA, they are in the same plane, and therefore the points B, P, C, are in the same straight line; and as BOC is a right angle, the rectangle under BP and PC is equal to the square of the perpendicular OP; but QOR is also a right angle, and therefore $QP \times PR = OP^2$; whence $BP \times PC = QP \times PR$, and therefore the points B, R, C, Q, are in the circumference of a circle, so that the angle at C is a right angle, being in the same segment with the angle at R. Thus the point C describes the circle whose diameter is PQ, and OC describes the cone of which this circle is the section.

8. Of the two right lines OP and OQ perpendicular to the given planes, one is also perpendicular to the plane of the section. That one is OP. Its extremity P is the focus of the parabola. The extremities of both are the extremities of the diameter PQ of the circle. The vertex of the parabola is the point R where the diameter of the circle intersects that given plane to which the plane of section is *not* parallel.

9. Theorem III. In a straight line at right angles to any diametral section QOq of an ellipsoid abc whose centre is O, let OT and OV be taken respectively equal to OQ and Oq the semi-axes of the section, and imagine the double surface which is the locus of all the points T and V; then if OS be perpendicular to the plane which touches the surface in T, and OP to the plane which touches the ellipsoid in Q, the lines OP and OS will be equal and perpendicular to each other, and the four straight lines OP, OQ, OS, OT, will lie in the same plane at right angles to Oq.

10. This theorem is taken from a former communication to the Academy*. The surface to which it relates, being the *wave surface* of Fresnel, is one of frequent occurrence in optical inquiries, and it is therefore desirable to give it a distinctive name not derived from any physical hypothesis. I shall call it a *biaxal surface,* from the circumstance implied in its construction, and adopted as the definition on which the preceding theorem is founded;—namely, that any pair of its coincident diameters are equal to the *two axes* of a central section made in the *generating ellipsoid abc*, by a plane perpendicular to the common direction of the two diameters. The name, perhaps, may appear the more appropriate, as it reminds us of the place which the surface holds in the optical theory of biaxal crystals.

11. Theorem IV. The biaxal surfaces generated by two reciprocal ellipsoids are themselves reciprocal.

For if Q and R (*Fig.* 4.) be reciprocal points on the two ellipsoids, abc and $a'b'c'$, a tangent plane at Q will cut OR perpendicularly in P; a tangent plane at R will cut OQ perpendicularly in N; and the rectangles ROP and NOQ will be equal to each other and to k^2 (*Art.* 4). Also if the straight line Oqr, at right angles to the plane of the figure, cut the first ellipsoid in q and the second in r, then (5) the elliptic section QOq will have OQ and Oq for its semi-axes, and the lines OR and Or will be the semi-axes of the other section ROr. Draw therefore, in the plane of the figure, the right lines OTL and OSM perpendicular to the right lines OQN and OPR, making OT, OL, OS, OM, equal to OQ, ON, OP, OR, respectively; the angles at S and L being of course right angles. Then it is evident that the point T is on the biaxal surface generated by the ellipsoid abc, because OT is perpendicular to the plane of the ellipse QOq and equal to the semi-axis OQ; and by Theorem III. it appears that OS is perpendicular to the tangent plane at T. In like manner, the point M is on the biaxal surface generated by the other ellipsoid $a'b'c'$, and OL is perpendicular to the tangent plane at M. Moreover, the rectangles MOS and LOT, being equal to the rectangles ROP and NOQ, are each equal to k^2. Hence the proposition is manifest.

12. As the ellipsoid whose semi-axes are $a, b, c,$ may be called the ellipsoid abc, so the biaxal surface generated by this ellipsoid may be called the biaxal abc; and that which is generated by the ellipsoid $a'b'c'$ may be called the biaxal $a'b'c'$.

13. Proposition V. To find what properties of biaxal surfaces are indicated by the cases wherein one of the two sections QOq, ROr, in the preceding theorem, is a circle.

Case 1. When QOq is a circular section of the ellipsoid abc, the points T and V, (9) in the description of the biaxal surface abc, coincide in a single point n. At this

* Transactions of the Royal Irish Academy, Vol. XVI. Part II. pp. 67, 68.

point there are an infinite number of tangent planes; because the semi-axes of the circular section QOq being indeterminate, any two perpendicular radii of the circle may take the place of OQ, Oq, in the general construction. The point n is therefore a point of intersection (3), where the two biaxal sheets cross each other, and it may be called a *nodal point*, or simply a *node*. As OQ always lies in the plane of the circle QOq, the line OR, which is reciprocal to OQ, must lie (6) in a given plane reciprocal to the plane of the circle. And as Oq lies in the plane of the circle, we have three right lines OR, Oq, OS, which are at right angles to each other, and of which the first two are confined to given planes. Therefore by Theorem II. the third line OS describes a cone whose sections parallel to the given planes are circles. Now TS—or in the present case nS—is parallel to the fixed plane which contains OR, and therefore the point S describes a circle; or, in other words, the feet of the perpendiculars OS, let fall from O on the nodal tangent planes, occupy the circumference of a circle passing (8) through the nodal point.

14. Parallel to the plane of the circle and to its reciprocal plane, conceive two planes passing through the node, and call them the *principal tangent planes* at n. The plane of the circle and its reciprocal plane are intersected in the right lines Oq, OR, by the plane qOR which is parallel to a tangent plane at n. Consequently this tangent plane at n intersects the two principal tangent planes in lines that are parallel to Oq, OR; and as Oq, OR are perpendicular to each other, it follows that every nodal tangent plane intersects the two principal tangent planes in lines that are at right angles.

Hence again, the nodal tangent planes touch (7) the surface of a cone whose sections, parallel to the principal tangent planes, are parabolas. As this cone touches the biaxal surface all round the point n, it may be called the *nodal tangent cone*.

15. *Case* 2. When ROr is a circular section of the ellipsoid $a'b'c'$, any two perpendicular radii of the circle may be taken for OR, Or: and because $OR=b'$, and $OR\times OP=k^2=bb'$, we have OP or OS equal to b, the mean semiaxis of the ellipsoid abc. Hence OS is given both in position and length; for it is perpendicular to the fixed plane ROr, and it is equal to b. Now a plane cutting OS perpendicularly at S, is a tangent plane to the biaxal abc; and we have just seen that this tangent plane remains the same, whatever pair of rectangular radii are taken for OR, Or. But the point of contact T is variable, for the plane ROS in which it lies changes with OR. Therefore as OR revolves, the point T describes a *curve of contact* on the tangent plane of the biaxal abc.

The lines OR, Or, are in the fixed plane ROr; and as OQ is reciprocal to OR, it lies in a fixed plane reciprocal to the plane Ror (6). Therefore the first two of the three perpendicular right lines Or, OQ, OT, are confined to fixed planes. Hence the third line OT describes a cone, whose sections parallel to these planes are circles. But the tangent plane is parallel to the fixed plane ROr, and its intersection with

OT describes the curve of contact. Therefore the curve of contact is a circle passing (8) through the point S.

16. We have examined the two cases of circular section with reference only to the biaxal abc. If we examine the same cases with regard to the second biaxal $a'b'c'$, we shall find that their indications are reversed; the supposition which gives a node upon one biaxal, giving a circle of contact on the other: and that the node and the circle, thus corresponding, are so related, that a line drawn from O to the node passes through the circumference of the circle, cutting the plane of the circle perpendicularly; whilst every line drawn from O through the circumference of the circle is perpendicular to some nodal tangent place.

These things are evident on looking at the figure. For when ROr is a circle, it is plain that the point M is a node of the biaxal $a'b'c'$, since OM is perpendicular to the plane of the circle ROr and equal to its radius OR. But we have already seen (15) that when ROr is a circle, the other biaxal abc has a circle of contact, whose plane is perpendicular to OM at the point S of its circumference. The line OTL is perpendicular, in general (11), to a tangent plane at M, and therefore perpendicular, in the present case, to a nodal tangent plane; whilst the point T, through which it passes, is on the circle of contact. It is also evident that $OT \times OL = k^2$.

We have here an example of the general remark in the corollary of Theorem I.

17. The section made in a biaxal surface abc, by any of the principal planes of its generating ellipsoid, consists of an ellipse and a circle.

For let the plane QOq pass through one of the semiaxes a, and let it revolve round this semiaxis, while the right line OTV (9), perpendicular to the plane QOq, revolves about O in the plane of the semiaxes b,c. Then the semiaxis a of the ellipsoid will always be one of the semiaxes of the ellipse QOq; and if OT be equal to this semiaxis, the point T will describe a circle with the radius a about the centre O. The other semiaxis of the ellipse QOq is that semidiameter of the principal ellipse bc which lies in the intersection of the plane bc with the plane QOq; and as OV is equal and perpendicular to this semidiameter, the point V describes an ellipse equal to bc, but turned round through a right angle, so that the greater axis of the ellipse described by V coincides in direction with the less axis of the ellipse bc. As the radius a of the circle is greater (4) than both the semiaxes b,c, of the ellipse, the circle will lie wholly without the ellipse.

In like manner, the section made in the biaxal surface by the plane ab consists of a circle with the radius c, and an ellipse with the semiaxes a,b; and as the radius of the circle is less than both the semiaxes of the ellipse, the circle lies wholly within the ellipse.

18. But when the section lies in the plane of the greatest and least semiaxes a,c, the circle and ellipse, of which it is composed, intersect each other. For the radius b of the circle is less than one semiaxis of the ellipse ac and greater than the other.

Leaving the ellipse ac in the position which it has as a section of the ellipsoid abc, if we describe the circle b with the centre O and radius b, the ellipse and the circle will cut each other in four points at the extremities of two diameters; and planes, passing through these diameters and through the semiaxis b of the ellipsoid, will evidently be the planes of the two circular sections of the ellipsoid. Now turning the ellipse ac round through a right angle (17), the circle and the ellipse in its new position will constitute the section of the biaxal surface, and will cut each other (*Fig.* 5.) in four points n at the extremities of two diameters nOn, nOn, which are perpendicular to the two former diameters, and therefore perpendicular to the planes of the two circular sections. Consequently, the biaxal surface has four nodes at the four points n. These nodes, it is manifest, are alike in all their properties; and they are the only points common to the two biaxal sheets, since the points T and V (9), in the description of the biaxal surface, cannot coincide unless the section QOq, perpendicular to OTV, be a circle.

19. The plane of the greatest and least semiaxes a, c, of the generating ellipsoid, may be called the plane of the nodes; and the two diameters nOn, nOn, passing through the nodes, may be called the *nodal diameters*.

At one of the nodes n (*Fig.* 5) draw tangents nf, nk, to the ellipse and the circle that compose the biaxal section; and through O draw Op perpendicular to On, cutting the circle in p. Then as On is perpendicular to the plane of a circular section of the ellipsoid abc, this circular section will have Op for its radius, and its circumference will cross that of the ellipse ac (belonging to the ellipsoid) in the point p. A line touching the ellipse ac at p will be parallel to every plane that touches the ellipsoid in a point of the circular section, and will therefore (6) be perpendicular to the plane which is reciprocal to the plane of the circular section. But the tangent at p is perpendicular to the tangent nf, since the two tangents would coincide if the ellipse ac were turned round (18) through a right angle, the point p then falling upon n. Hence the circular section and its reciprocal plane are parallel to the tangents nk, nf; and therefore two planes perpendicular to the plane of the figure and passing through these tangents, are the planes that we have called (14) the principal tangent planes at n.

20. Produce Op to meet nf in v, and conceive a parabola having its focus at O, its vertex at v (8), and its plane perpendicular to the plane of the figure. A cone, with its vertex at n and this parabola for its section, is (14) the nodal tangent cone.

Draw Of perpendicular to nf at f, and meeting nk in k. The perpendiculars let fall from O upon the nodal tangent planes form a cone, of which the circles described in planes perpendicular to the figure upon the diameters nf, nk, are sections (8). On the other biaxal surface $a'b'c'$ there is (16) a circle of contact whose plane is perpendicular to On. This circle of contact is (16) another section of the cone last mentioned.

21. To the circle b and to the principal section ac of the ellipsoid abc conceive a

common tangent $d'i'$ to be drawn, in a quadrant adjacent to that which contains the node n, and let it touch the circle in d' and the ellipse ac in i'. A radius Od'', drawn through the point d' to meet the ellipsoid $a'b'c'$ in the point d'', will be reciprocal to the radius Oi', because it is perpendicular to a tangent at i', and it will be equal in length to b', because $Od'' \times Od' = k^2 = bb'$, and $Od' = b$; whence $Od'' = b'$. Therefore Od'' is in a circular section of the ellipsoid $a'b'c'$. Two planes perpendicular to the plane of the figure, and passing through the reciprocal radii Od'', Oi', are (6) reciprocal planes, and we have seen that the first of them makes a circular section in the ellipsoid $a'b'c'$. They are therefore (15) the fixed planes in the second case of Prop. V.

22. Now draw di a common tangent to the circle b and ellipse ac composing the biaxal section, and let it touch the circle in d and the ellipse in i. The lines Od, Oi, are of course perpendicular to the lines Od', Oi', and therefore perpendicular to the fixed planes just mentioned. Hence the line Od and the point d are the same as the fixed line OS and the point S in the second case of Prop. V. The plane of the circle of contact is therefore perpendicular to Od at the point d (15); and the points d and i, where its plane intersects the right lines Od, Oi, perpendicular to the fixed planes, are (8) the extremities of a diameter.

These things agree with the obvious remark, that the points of contact d and i must be points of the circle of contact; and that di must be a diameter, because the plane of the circle is perpendicular to the plane of the figure, and this latter plane divides the biaxal surface symmetrically.

As the circle and ellipse may have a common tangent opposite to each node, there are four circles of contact in planes perpendicular to the plane of the nodes.*

23. The biaxal surface belongs to a class that may be called *apsidal surfaces*, from the manner in which they are conceived to be generated.

Let G be a given surface, and O a fixed origin or pole. If a plane passing through O cut the surface G, the curve of intersection will in general have several apsides A, A', A'', &c., where the lines OA, OA', OA'', &c. are perpendicular to the curve. Through the point O conceive a right line perpendicular to the plane of the curve, and on this perpendicular take from O the distances Oa, Oa', Oa', &c. respectively equal to the apsidal distances, OA, OA', OA'', &c. Imagine a similar construction to be made in every possible position of the intersecting plane passing through O, and the points a, a', a'', &c. will describe the different sheets of an *apsidal surface*.

* The curves of contact on biaxal surfaces, and the conical intersections or nodes, were lately discovered by Professor Hamilton, who deduced from these properties a theory of conical refraction, which has been confirmed by the experiments of Professor Lloyd. See Transactions of the Royal Irish Academy, Vol. XVII. Part. I, pp. 132, 145; and the present paper, Art. 55—58.

The indeterminate cases of circular section—at least the case of the nodes—had occurred to me long ago; but having neglected to examine the matter attentively, I did not perceive the properties involved in it (13). *April* 2, 1834.

The apsidal surface has a centre at the point O, because the lengths Oa, Oa', Oa'', &c. may be measured on the perpendicular at either side of the intersecting plane.

Referring* to the demonstration of Theorem III. it will be seen to depend only on the supposition that the point Q is an apsis of the section made by the plane QOq; or, which is the same thing, that OQ is a position wherein the radius rector from O to the curve of section is a *maximum* or a *minimum*. Hence we have the following general theorem :—

24. Prop. VI. Theorem. If tangent planes be applied at corresponding points A, a, on the surface G and the apsidal surface which it generates; these tangent planes will be perpendicular to each other and to the plane of the points O, A, a.

This is equivalent to saying that perpendiculars from O on the tangent planes are equal to each other, and lie in the plane of the lines OA, Oa.

25. If Q and R be reciprocal points on two reciprocal surfaces of which O is the fixed origin or pole, the tangent plane at Q will be (1) perpendicular to OR and to the plane QOR. Let a plane also perpendicular to the plane QOR pass through OQ, cutting the surface to which the point Q belongs in a certain curve, and the tangent plane at Q in a tangent to this curve. The tangent is evidently perpendicular to OQ, and therefore the point Q is an apsis of the curve.

In like manner, the point R is an apsis of the section made in the other surface by a plane passing through OR and perpendicular to the plane QOR.

26. From these observations, and from Prop. VI., it appears that if the points Q, R, in the figure of Theorem IV., be reciprocal points on *any* two reciprocal surfaces, and if the same construction be supposed to remain, the points T and M will be points on the apsidal surfaces generated by these reciprocal surfaces, and the tangent planes at T and M will be perpendicular to the lines OM and OT respectively. Also the rectangles LOT and MOS will be equal to k^2. Hence we have another general theorem :—

Prop. VII. Theorem. The apsidal surfaces generated by two reciprocal surfaces are themselves reciprocal.

27. A very simple example of apsidal surfaces, with nodes and circles of contact, may be had by supposing the generatrix G to be a sphere, and the pole O to be within the sphere, between the surface and the centre C.

It is evident that the apsidal surface in this case will be one of revolution round the right line OC as an axis. Therefore taking for the plane of the figure (*Fig.* 6.) a plane passing through OC and cutting the sphere in a great circle of which the radius is CS, let a plane at right angles to the figure revolve about O, cutting the circle CS in the points A, A'. The section of the sphere made by the revolving plane will have only

* Transactions of the Royal Irish Academy, Vol. XVI. Part II. p. 68.

two apsides A, A', with respect to the point O, except when the plane is perpendicular to OC. Hence if we draw the right line Oaa' perpendicular to AOA', taking Oa, Oa', always equal to OA, OA', the points a, a', will describe a section of the apsidal surface. This section will evidently consist of two circles $C'S'$, $C''S''$, equal to the circle CS, and having their centres C', C'', on the opposite sides of O in a right line $C'OC''$ perpendicular to OC; the distances OC, OC', OC'' being equal. The circles $C'S'$, $C''S''$, intersect in two points n, n', on the line OC and have two common tangents di, $d'i'$, which are bisected at right angles by OC in the points c,c'.

28. Now let the circles $C'S'$, $C''S''$, with their common tangents, or only one of the circles with the half tangents, revolve about the axis OC, and we shall have the apsidal surface with nodes at n,n', and with circles of contact described by the radii cd, $c'd'$.

The section of the sphere by a plane passing through O at right angles to On, is a circle of which O is the centre. If therefore we suppose that the point n answers to a in Prop. VI., the apsis A corresponding to n will be indeterminate, and the position of the tangent plane at n will also be indeterminate, which ought to be the case at a node.

The surface reciprocal to the sphere, the pole being at O, is evidently a surface of revolution about the axis OC (it is easily shown to be a spheroid having a focus at O); and the section of this reciprocal surface, by a plane perpendicular to the axis at O, is a circle of which O is the centre. This circumstance indicates (15) that on the apsidal surface there is a curve of contact, whose plane is parallel to the plane of circular section; which agrees with what we have already seen.

29. When the point O is without the sphere, the axis OC will pass between the circles $C'S'$, $C''S''$, without intersecting either of them. The apsidal surface, described by the revolution of one of these circles about OC, will be a circular ring. The nodes have disappeared; but the circles of contact still exist, as is evident.

Part II.—On the Wave Theory of Light.

30. Some of the foregoing propositions lead to a simple transformation of the wave theory of light.

In this theory, the *surface of waves*, or the *wave surface*, is a geometrical surface used to determine the directions and velocities of refracted or reflected rays; being the surface of a sphere in a singly refracting medium; a double surface, or a surface of two sheets, in a doubly refracting medium; a surface of three sheets on the supposition of triple refraction; and having always a centre O round which it is symmetrical. The radii of the wave surface, drawn from its centre O in different directions, represent the velocities of rays to which they are parallel.

31. We shall consider particularly the case of a doubly refracting crystal, with two plane faces parallel to each other, and surrounded by a medium of the common kind wherein the constant velocity is V; supposing, for the sake of clearness, that the crystal refracts more powerfully than the surrounding medium, so that the velocities in the crystal are less than the velocity V.

A ray $S'O$, falling on the first surface of the crystal at the point O, is partly reflected according to the common law of reflection, and partly refracted. The two refracted rays pass on to the second surface, where each of them is divided by internal reflection into a pair, the two reflected pairs being parallel to each other; while the two emergent rays—one from each refracted ray—are parallel to each other and to the incident ray $S'O$. The directions of the rays within the crystal are usually found by the following construction.

32. Describe a wave surface of the crystal, having its centre at O the point of incidence. By the nature of the wave surface, a right line OTU, drawn from the point O, will in general cut this surface in two points T, U, on the same side of O; and a ray passing through the crystal in a direction parallel to OTU will have one of the two velocities represented by the radii OT, OU, taking a line of a certain length k to represent the uniform velocity V in the external medium. With the centre O and a radius OS equal to this line k describe a sphere. As the velocities in the crystal are supposed to be less than V, the wave surface will lie wholly within this sphere. Let the plane of the figure (*Fig.* 7) be the plane of incidence, perpendicular to the parallel faces of the crystal, and intersecting the first face in the right line FA. Through the point S, where the incident ray $S'O$, produced through the crystal, cuts the surface of the sphere, draw SI at right angles to OS and meeting FA in the point I. A right line perpendicular to the plane of the figure, and passing through this point I, we shall call the right line I.

33. Through the right line I draw two planes touching the two sheets of the wave surface, on the side remote from the incident light, in the points T, T', which will lie within the sphere (32); then the incident plane wave, perpendicular to OS, will be refracted into two plane waves parallel to these two tangent planes; and the lines OT, OT', will be the directions of the refracted rays along which the refracted waves are propagated. The lengths OT, OT', represent the velocities with which the light moves along the rays; and of course the normal velocities, which are the velocities of the refracted waves, are represented by the perpendiculars OG, OH, let fall from O on the two tangent planes at T, T'. These two perpendiculars OG, OH, evidently lie in the plane of the figure; but the points T, T', in general, do not lie in this plane.

34. Again, through the right line I draw two other planes touching the wave surface, at the side of the incident light, in the points t, t'. The rays OT, OT', arriving at the second surface of the crystal, will each be divided by internal reflection into two rays parallel to Ot, Ot'; and these four reflected rays, arriving at the first surface, will each be divided,

by a new reflection, into two rays parallel to OT, OT'; and so on, for any number of reflections. Any of the rays emerging at the first surface after internal reflections, is parallel to the ray Os produced by ordinary reflection at the point of incidence; and any ray emerging at the second surface is parallel to the incident ray $S'OS$.

35. This construction may be changed into another that will be found more convenient both in theory and practice.

Through S draw SR perpendicular to OI, and meeting OG, OH, produced, in the points P, M. Then as the angles at G and R are right angles, the points I, R, G, P, are in the circumference of a circle, and therefore $OP \times OG = OI \times OR = OS^2 = k^2$; and similarly, $OM \times OH = k^2$. If then we take O for the fixed origin, or pole, and k^2 for the constant rectangle (Theorem I.), and describe the surface which is reciprocal to the wave surface, it is evident that the points P and M will be points of the surface so described, and that OT, OT', will coincide in direction with perpendiculars let fall from O on planes touching the surface at P and M, and will be inversely proportional to these perpendiculars. It follows in the very same manner, that if perpendiculars Og, Oh, let fall from O on the tangent planes at t, t', be produced to meet SR in the points p, m, these points will also be on the surface reciprocal to the wave surface.

In the present case, it is manifest that this reciprocal surface lies wholly without the sphere OS.

36. The surface reciprocal to the wave surface, the pole being at O, we shall call the *surface of refraction*.

It is hardly necessary to observe that the surface of refraction has a centre at the point O, round which it is symmetrical; that it is a sphere in a singly refracting medium, a double surface in a doubly refracting medium, and a surface of three sheets if we suppose a case of triple refraction.

37. In the case that we are considering, let the figure (*Fig.* 8.) represent a section made in the double surface of refraction and its attendant sphere by the plane of incidence. Through the point S, where the incident ray $S'O$ prolonged cuts the circular section of the sphere, draw SR perpendicular to the face of the crystal, or to FA; and let SR produced cut the circle again in the point s. Then Os is the direction of the ray given by ordinary reflection at the first surface of the crystal. Produce the right line SRs both ways, to cut the surface of refraction in the points P, M, behind the crystal, and in the points p, m, before it; and conceive planes to touch the surface of refraction at the points P, M, p, m. Suppose also that perpendiculars OP', OM', Op', Om', are let fall from O upon these tangent planes, and that they intersect the planes in the points P', M', p', m', respectively.

Then from the preceding observations (33, 34, 35), it is manifest that OP', OM', are the directions of the rays into which $S'O$ is divided by refraction; that each of these refracted rays, on arriving at the second surface of the crystal, is divided by in-

ternal reflection into two rays parallel to Op', Om'; and that each of the four reflected rays, on arriving at the first surface, is again divided by reflection into two rays parallel to OP', OM'; and so on. In general, every ray going into the crystal from the first surface, whether after refraction or after any even number of internal reflections, is parallel either to OP' or to OM'; and every ray returning from the second surface of the crystal after any odd number of internal reflections, is parallel either to Op' or to Om'. Thus the direction of every ray in the interior of the crystal is the same as the direction of some one of the four lines OP', OM', Op', Om'; and the velocity of the ray is inversely as the length of this line; so that the velocity of the ray OM', for example, or of any ray parallel to OM', is to the velocity V as OS is to OM'. The little plane waves that, keeping always parallel to themselves, move along these rays, are respectively perpendicular to the lines OP, OM, Op, Om; and the lengths of these lines are inversely as the velocities of the waves estimated in directions perpendicular to their planes; so that the velocity of the wave which moves along the ray OM', or along any parallel ray, is to the velocity V as OS is to OM.

38. The ray OP' and all the rays parallel to it are perpendicular to the plane which touches at P the surface of refraction; and the waves which move along these rays are perpendicular to the right line OP. Any ray of this set may be called a *ray* P, and any of the waves a *wave* P. In like manner, the *rays* M, p, m, are rays that are perpendicular to the tangent planes at the points M, p, m, respectively; and the *waves* M, p, m, are the waves that belong to these rays, and that have their planes respectively perpendicular to the right lines OM, Op, Om. The rays P, M, all come from the first surface of the crystal; the rays p, m, from the second.

As the ordinates RP, Rp, are greater than the ordinates RM, Rm, so the rays P, p, are more refracted or more reflected than the rays M, m. The former rays may therefore be said to be *plus refracted*, or *plus reflected*, and the latter to be *minus refracted*, or *minus reflected*. Or,—for the convenience of naming,—the rays P, p, may be called *plus rays*; and the rays M, m, *minus rays*. The waves P, p, in like manner, may be termed *plus waves*, and the waves M, m, *minus waves*.

For a medium of the common kind, or a singly refracting medium, we may use the letters S and s. Thus the incident ray $S'OS$, or any ray emerging parallel to OS from the second surface of the crystal, may be marked by the letter S; while the ray Os produced by common reflection, or any ray emerging parallel to Os from the first surface, may be denoted by the letter s.

39. The course of a ray through the crystal may now be easily expressed. A ray $SMps$, for example, is a ray (S) incident on the crystal, undergoing minus refraction (M) at the first surface, plus reflection (p) at the second, and emerging (s) from the first surface in a direction parallel to Os. Of this ray the part within the crystal is Mp. A ray SPS is a ray plus refracted, and then emerging in a direction parallel to that of incidence. A ray $SPpMS$ is a ray plus refracted at the first surface, then

plusreflected at the second surface, then minus reflected at the first surface, and finally emerging from the second surface in a direction parallel to that of incidence. Its path within the crystal is ***PpM.***

These examples indicate the general method of expressing the path of a ray.

40. Suppose light to be moving in the same direction and with the same velocity along two proximate parallel rays, so that it is at the point A in one ray when it is at the point B in the other; and through the points A and B conceive two planes perpendicular to the common direction of the rays. These planes are either coincident, or maintain a constant distance. In the first case, the rays are said to be in complete accordance. In the second case, the constant distance between the planes is called the *interval* between the portions of light composing the rays, or the interval between the waves that move along the rays.

We proceed to find the lengths of these intervals in the case of rays emerging parallel to each other, at either side of the crystal that we have been hitherto considering.

41. Let the tangent planes at P, M, p, m, intersect the plane of the figure (*Fig.* 8) in the right lines $PP_{,}$, $MM_{,}$, $pp_{,}$, $mm_{,}$, which of course are tangents to the section of the surface of refraction represented in the figure; let a perpendicular at O to the face of the crystal cut these tangents in the points $P_{,}$, $M_{,}$, $p_{,}$, $m_{,}$; and let the lines OP'', OM'', Op'', Om'', respectively parallel to $PP_{,}$, $MM_{,}$, $pp_{,}$, $mm_{,}$, cut the line SRs in the points P'', M'', p'', m''.

The length of the path which a ray P describes within the crystal, is equal to the thickness Θ of the crystal divided by the cosine of the angle $P'OP_{,}$ which the path of the ray makes with a perpendicular to the faces of the crystal; and the velocity of P is equal to $V \times \frac{OS}{OP'}$ (37); dividing therefore the length of the path by the velocity, we find that the time in which a ray P crosses the crystal is equal to $\frac{\Theta \times OP'}{V \times OS \times \operatorname{Cos} P'OP_{,}}$. But as OP' is perpendicular to the tangent plane at P, we have $\frac{OP'}{\operatorname{Cos} P'OP_{,}} = OP_{,} = PP''$. Therefore the time is equal to $\frac{\Theta \times PP''}{V \times OS}$. Similarly, the times in which rays M, p, m, pass from one surface of the crystal to the other, are equal to $\frac{\Theta \times MM''}{V \times OS}$, $\frac{\Theta \times pp''}{V \times OS}$, $\frac{\Theta \times mm''}{V \times OS}$, respectively.

42. Now suppose the path of a ray P to be be projected perpendicularly on a right line having any proposed direction in space. Through O conceive a right line OL parallel to the proposed direction, and meeting in L the tangent plane at P. The length of the projection is equal to the length of the path multiplied by the cosine of the angle $P'OL$ which the ray P makes with OL; that is, the projection is equal to $\Theta \frac{\operatorname{Cos} P'OL}{\operatorname{Cos} P'OP_{,}}$. But because OP' is perpendicular to the tangent plane at P, we have $\operatorname{Cos} P'OL = \frac{OP'}{OL}$, and $\operatorname{Cos} P'OP_{,} = \frac{OP'}{OP_{,}} = \frac{OP'}{PP''}$; therefore $\frac{\operatorname{Cos} P'OL}{\operatorname{Cos} P'OP_{,}} = \frac{PP''}{OL}$. Hence the projection is equal to $\Theta \frac{PP''}{OL}$.

If the path of a ray P be projected on the incident ray OS, then producing OS to meet $PP_{,}$ in l, we see, by what has just been proved, that the length of the projection is equal to $\Theta \frac{PP''}{Ol} = \Theta \frac{SP''}{OS}$, by similar triangles. In like manner, the projections of the paths of rays M, p, m, on the direction of the incident ray OS, are equal to $\Theta \frac{SM''}{OS}$, $\Theta \frac{Sp''}{OS}$, $\Theta \frac{Sm''}{OS}$, respectively.

43. Let each rectilinear path be measured in the direction in which the light moves along it; and according as the direction so measured makes an acute or an obtuse angle with the direction OS, measured from O to S, let the projection of the path on OS be reckoned positive or negative. Then if $SPmMpMS$ be any ray entering the crystal at O, and emerging from its second surface at E, and if a perpendicular EI be let fall from E upon OS, meeting OS in I; the distance OI, from O to to the foot of this perpendicular, will evidently be equal to the algebraic sum of the projections of the paths P, m, M, p, M, contained within the crystal; taking each projection with its proper sign. It is obvious that the projections of the P and M rays are always positive. And as the lines Op', Om',—the directions of the rays p, m,—lie in planes which are respectively perpendicular to $pp_{,}$, $mm_{,}$, or to Op'', Om'', it is easy to see that these directions make acute or obtuse angles with OS, according as the points p'', m'', lie below the point S or above it; that is, the projections are positive or negative according as the points p'', m'', lie without the circle OS towards P, M, or within the circle. Therefore the distance OI, in the case of the figure, is equal to $\frac{\Theta}{OS}(SP''—Sm''+SM''—Sp''+SM'')$.

44. If the paths of rays P, M, p, m, be projected on the direction Os of the ordinarily reflected ray, the lengths of their projections will be $\Theta \frac{sP''}{OS}$, $\Theta \frac{sM''}{OS}$, $\Theta \frac{sp''}{OS}$, $\Theta \frac{sm''}{OS}$, respectively. The projections upon Os of the rays p, m, will be always positive; and the projections of the rays P, M, will be positive or negative according as the points P'', M'', lie above the point s or below it; that is, according as the points P'', M'', lie without the circle OS towards p and m, or within the circle. So that if $SPmMps$ be a ray entering the crystal at O and emerging from the first surface at e, and if a perpendicular ei be let fall from e upon Os, the distance Oi from the point O to the foot of this perpendicular, or the algebraic sum of the projections of the paths P, m, M, p, contained within the crystal, will be equal to $\frac{\Theta}{OS}(—sP''+sm''—sM''+sp'')$, in the case of the figure.

45. Let us imagine that the light in the incident ray $S'O$, instead of being interrupted at O by the crystal, had continued to move with the same velocity V in the same right line OS, leaving the point O at the moment when the refracted light enters the crystal at O. Comparing the light in this imaginary ray with that in a ray emerging parallel to it from the second surface of the crystal, after an even number of internal reflections, we shall find that the emergent is behind the imaginary ray, and

that the interval between them (40),—or the retardation of the former,—may be derived very easily from the letters that designate that ray. Let $SPmMpMS$ be any such ray. The sum of the distances of the point S from each of the points marked by the letters ($PmMpM$) that denote (39) the part of the ray contained within the crystal, is proportional to the interval of retardation; that interval being equal to $\frac{\Theta}{OS}(SP+Sm+SM+Sp+SM)$.

For if from the point E, where the last internal ray M emerges from the second surface of the crystal, a perpendicular EI be let fall upon OS, meeting OS in I, the time of describing OI with the velocity V would (43) be $\frac{\Theta}{V\times OS}(SP''-Sm''+SM''-Sp''+SM'')$. But (41) the actual time of describing the broken path $PmMpM$ is $\frac{\Theta}{V\times OS}(PP''+mm''+MM''+pp''+MM'')$; and, on inspecting the figure, this time is seen to be greater than the time of describing OI, by $\frac{\Theta}{V\times OS}(SP+Sm+SM+Sp+SM)$, or by the time in which the line $\frac{\Theta}{OS}(SP+Sm+SM+Sp+SM)$ would be described with the velocity V. Consequently, at the moment when the light in the ray $SPmMpMS$ emerges at the point E from the second surface of the crystal, the light in the imaginary uninterrupted ray OS will have passed the point I by an interval equal to the line just mentioned; and as the two rays afterwards have the same velocity and parallel directions, this interval is the retardation of the emergent ray.

46. The rays emerging from the first surface after any odd number of internal reflections are to be compared with the ordinarily reflected ray Os to which they are parallel; the light in Os, which moves with the velocity V, being supposed to leave O at the moment when the refracted light enters the crystal at O. The mode of proceeding in this case is exactly similar to that in the last, and the interval is determined in the same way, using s in place of S; the retardation of the ray $SPmMps$, for example, of which the part $PmMp$ is contained within the crystal, being equal to $\frac{\Theta}{OS}(sP+sm+sM+sp)$.*

47. It is remarkable that the preceding demonstration nowise depends upon the supposition that the planes perpendicular to the rays P, M, p, m, are tangent planes to the surface of refraction at the points P, M, p, m. If we had supposed any planes—different from the plane of the figure— to pass through the points P, M, p, m, and the rays to coincide in direction with perpendiculars let fall from O upon these planes, and to have velocities inversely proportional to the lengths of the perpendiculars, the intervals of retardation would have remained unchanged. Hence the retardations are the same as if the lines OP, OM, Op, Om, were the directions of the rays in passing

* The *change of phase*, which may take place at a surface of the crystal, is not here considered as affecting the intervals.

through the crystal; as will appear by conceiving the planes that we have spoken of to be perpendicular to these lines.

If the incident ray $S'O$ were refracted in the ordinary way with an index equal to $\frac{OP}{OS}$, it would take the direction OP; if it were refracted, in like manner, with the index $\frac{OM}{OS}$, it would take the direction OM; and if the two rays, thus ordinarily refracted, were to emerge from the second surface of the crystal in directions parallel to OS, it is evident from what has been said, that they would be in complete accordance, respectively, with the rays SPS and SMS.

If the surface of refraction should happen to have a node N, which is a point of intersection where it admits an infinite number of tangent planes (3), let the direction of the incident ray $S'OS$ be chosen, so that the right line RS perpendicular to the face of the crystal, being produced below S, may pass through N, and we shall have a cone of refracted rays formed by the perpendiculars let fall from O upon the tangent planes at N; all of which rays, on emerging parallel to OS from the second surface of the crystal, will be in complete accordance with one another. For we have just seen that if the ray $S'OS$ were supposed to emerge after being refracted in the ordinary way with an index equal to $\frac{ON}{OS}$, it would be in complete accordance with any ray of the cone.

48. The interval between any two rays emerging at the same side of the crystal is the difference of their retardations. In taking the difference, the letters that are common to the names of the two rays may be left out. Thus the ray $SPmMS$ is behind the ray SPS by the interval $\frac{\Theta}{OS}(Sm+SM)=\frac{\Theta}{OS}Mm$. The line $\frac{\Theta}{OS}Pp$ is the interval between the rays SMS and $SMpPS$, or between the reflected ray Os and the ray $SPps$; and so on.

49. The retardations of the two refracted rays SPS and SMS, emerging without internal reflection, are $\frac{\Theta}{OS}SP$ and $\frac{\Theta}{OS}SM$ respectively. The difference of these is $\Theta\,\frac{PM}{OS}$. Consequently, when the two refracted rays have emerged from the second surface in directions parallel to the incident ray, the light in the plus emergent ray is behind the light in the minus emergent ray by an interval equal to $\frac{\Theta\times PM}{OS}$. Or, in other words, the incident plane wave, perpendicular to OS, produces two emergent waves parallel to each other and to the incident wave, moving along the emergent rays with equal velocities V, and preserving the distance $\frac{\Theta\times PM}{OS}$ between their planes, the minus wave being foremost. If OS, the radius of the sphere, be taken for unity, PM will be a number,—generally a very small fraction,—and the interval will be the thickness of the crystal multiplied by this number.

50. Suppose the right line PMR, remaining always perpendicular to the face of the crystal, to describe a cylindrical surface, with the condition that the part PM, inter-

cepted between the two sheets of the surface of refraction, shall remain of a constant length; the point R will then describe, on the surface of the crystal, a curve whose radii OR are the sines (to the radius OS) of the angles of incidence of a cone of rays; and every ray $S'O$ of this cone, when refracted by the crystal, will afford two emergent rays, or two waves, having the same given interval between them. Lines drawn from the eye parallel to the sides of this cone are the emergent rays belonging to a ring, when rings are made to appear, in any of the usual ways, on transmitting polarised light through the plate of crystal. In nominal conformity to this, we see that the line PM describes a ring of constant breadth between the two sheets of the surface of refraction. The ring described by supposing pm to remain constant corresponds to the interval between two rays p and m reflected at the same point of the second surface of the crystal, and then emerging at the first. The other intercepts Pp, Mm, Pm, Mp, are proportional (48) to intervals like those in Newton's rings; to the intervals, namely, between the reflected ray Os and the rays $SPps$, $SMms$, $SPms$, $SMps$, emerging at the first surface after one reflection within the crystal; or to the intervals between rays that are twice reflected in the crystal and the rays transmitted without reflection.

51. The general investigation of the figure of a geometrical ring does not distinguish between the different intercepts, and will therefore include all the rings PM, pm, Pp, Mm, Pm, Mp; so that it will be sufficient to contemplate any one of them, as PM, of which the breadth PM is equal to a given line I.

The points P and M describe, in general, similar and equal curves of double curvature, which may be called *ring-edges*, as being the edges of the ring; and if we imagine the surface of refraction, carrying these curves along with it, to be shifted either way, in a direction parallel to PM, through a distance equal to I, it is clear that the new position of one of the ring-edges will exactly coincide with the first position of the other, and that therefore the curve of the latter ring-edge will be given by the intersection of the two equal surfaces in these two positions. Let $U=0$,—where U is a function of x, y, z, and given quantities—be the equation of the surface of refraction in its original position; and, the axes of coordinates being fixed, suppose that by the shifting of the surface the coordinates of a point assumed on it are diminished by the given lines f, g, h, which are the projections of the given line I on the axes of x, y, z, respectively. Then the equation of the surface in its new position will be had by substituting $x+f$, $y+g$, $z+h$, for x, y, z, in the equation $U=0$, which will thus become $U+V=0$, where V is the increment of U produced by the substitution. These two equations combined are equivalent to the equations $U=0$, $V=0$, which are therefore the equations of one of the ring-edges. If the surface had been shifted the opposite way, in a direction parallel to PM, the intersection would have been the other ring-edge, whose equations are therefore deducible from those already found, by changing the signs of f, g, h.

52. If the equation of the surface of refraction be transformed, so that the plane of xy may coincide with the face of the crystal, and the axis of z be perpendicular to it, the origin of coordinates being at the centre O, no change will be produced in x or in y by the motion of the surface, because PM, the direction of the motion, is now parallel to the axis of z; but z will be diminished or increased by I; and accordingly, if $U'=0$ be the equation of the surface in its first position, when the centre is at O, and if U' become $U'+V'$ when z becomes $z+I$,—the equation of the surface in its second position, when the centre has moved through a distance equal to I along the axis of z, will be $U'+V'=0$; and these two equations combined will give $U'=0$, $V'=0$, for the equations of one of the ring-edges. The equations of the other ring-edge are deduced from these by changing the sign of I.

The projection of each of the ring-edges on the plane of xy is the curve traced by the point R on the surface of the crystal (50). This curve may be called a *ring-trace*. Its equation is obtained by eliminating z between the equations of a ring-edge; and as the result must be the same whether I be taken positive or negative, the equation of the ring-trace, when found by this general method, will contain only even powers of I. The radii drawn from O to the points R of the ring-trace, are (50) the sines (to the radius OS,) of the angles of incidence or emergence of the rays that form an optical ring; the rays that come from this ring to the eye being parallel to the sides of the cone described by the right line $S'OS$ while the point R describes the ring-trace.

53. It is evident that tangents to the ring-edges, at the points P and M, are parallel to each other, and therefore parallel to the intersection of two planes touching the surface of refraction at P and M, because these tangent planes pass through the tangents. But the directions OP', OM', are perpendicular to the tangent planes, and therefore the plane $P'OM'$, containing the two rays, is perpendicular to the intersection of the tangent planes, and of course perpendicular to the parallel tangents. Hence the plane $P'OM'$ intersects the face of the crystal in a right line perpendicular to the projection of the parallel tangents on the face of the crystal. As this projection is a tangent to the curve described by R, it follows that the normal to the ring-trace at the point R is parallel to the line joining the points in which the two refracted rays cut the second surface of the crystal.

In like manner, taking any two consecutive rays (P and m) having a common extremity on one surface of the crystal, the line joining the points where these rays cut the other surface, is parallel to the normal at the point R of the ring-trace which is described when the intercept (Pm) between the letters that mark the rays is supposed to remain constant.

54. In all that precedes we have made no supposition about the surface of refraction except that it is a surface of two sheets; and if we supposed it to have three sheets, the conclusions would be easily extended to this hypothesis.

In the theory of FRESNEL, the wave surface is* a biaxal whose generating ellipsoid has its centre at the point O, and its semiaxes parallel to the three principal directions of the crystal, the length of each semiaxis being equal to OS divided by one of the principal indices of refraction. The surface of refraction is reciprocal to the wave surface, and is (11) therefore another biaxal generated by an ellipsoid reciprocal to the former, having its centre at the same point O, and the directions of its semiaxes the same as before, the rectangle under each coincident pair of semiaxes being equal to k^2 or OS^2. Hence the semiaxes of the ellipsoid which generates the biaxal surface of refraction are equal in length to OS multiplied by each of the three principal indices. This biaxal surface is of course to be substituted for the surface of refraction in the preceding observations.

55. When the line RS, produced below S, passes through a node N of the biaxal surface of refraction, the points P, M, coincide in the point N, and the interval PM vanishes. At the point N there are an infinite number of tangent planes, and the perpendiculars from O on these tangent planes give a cone of refracted rays whose sections we have already shown how to determine (20). All the rays in this cone, on arriving at the second surface of the crystal, emerge parallel to the incident ray OS; and if the rays in the emergent cylinder be cut by a plane perpendicular to their common direction, they will all arrive at this plane at the same instant, because the interval PM vanishes. See art. 47.

56. Suppose fig. 5 to be a section of the wave surface. The right line Od will pass through N; and the circle of contact, described on the diameter di in a plane perpendicular to the right line OdN, will be a section of the refracted cone. Now it will be recollected† that, in general, the vibrations of a ray OT, which goes to any point T of the wave surface, are parallel to the line which joins the point T with the foot of the perpendicular let fall from O on the tangent plane at T. In the present case, the perpendicular is the same for all the rays of the refracted cone, and its extremity coincides with the point d: so that the line dT, drawn from d to any point T of the circle of contact, is parallel to the vibrations of the ray OT which passes through T. Conceive, therefore, a plane perpendicular to ON at the nodal point N. This plane will cut the refracted cone in a circle whose circumference will pass through N; and a line NT', drawn from the node to any other point T' of the circumference, will be the direction of the vibrations in a ray OT' which crosses the circle at this point. The plane of polarisation is perpendicular to the direction of the vibrations.

57. The transverse section of the emergent cylinder is always a very small ellipse, affording a hollow pencil of parallel rays in complete accordance (55). If the crystal be thin, this ellipse will be of evanescent magnitude. Hence the line OS will be the direction of a line drawn from the eye to the centre of the rings commonly observed

* Trans. R. I. A. Vol. XVI. p. 76. † Ibid.

(50) with polarised light; or it will be what is called the apparent direction of one of the *optic axes*. The diameter passing through N will be the direction of the optic axis within the crystal. There are therefore two optic axes, parallel to the two nodal diameters (19) of the surface of refraction.

As ON is equal to the mean semiaxis of the generating ellipsoid, or to the mean index of refraction, when OS is unity, it follows that the apparent direction of an optic axis is the direction of an incident ray, which, if refracted in the ordinary way, with an index equal to the mean index of refraction, would pass along a nodal diameter of the surface of refraction.

58. We have seen (15) that there is a circle of contact on the biaxal surface of refraction. If an incident ray $S'OS$ be taken, cutting the sphere in S, so that the line RS produced may pass through the circumference of this circle, it is manifest that the direction of the refracted ray will be the same through whatever point Π of the circumference the line RS may pass, because that direction is perpendicular to the tangent plane at Π, which is in fact the plane of the circle itself. If, therefore, the line RS move parallel to itself along the circumference of the circle, cutting the sphere in a series of points S, every incident ray $S'OS$ which passes through a point S so determined, will be refracted into two rays of which one will have a fixed direction in the crystal, being perpendicular to the plane of the circle of contact, and therefore coinciding (16) with nOn, one of the nodal diameters of the wave surface. But though the direction On of the refracted ray is fixed, its polarisation changes with the incident ray from which it is derived; for if Π be the point in which the line RS, corresponding to any position of the incident ray, crosses the circle of contact, the vibrations of the refracted ray On will be contained in the plane of the lines On, $O\Pi$, and will be perpendicular to $O\Pi$. Conceive a circle described on the diameter nf in a plane perpendicular to the figure (*Fig.* 5). This circle, and the circle of contact on the surface of refraction, are (20) sections of the same cone. Let Π' therefore be the point at which $O\Pi$, in any position of the incident ray, crosses the circumference of the circle nf; and the line $\Pi'n$, drawn to the node of the wave surface, will be the corresponding direction of the vibrations in the ray On.

59. With regard to the general law of polarisation in the theory of Fresnel, it may be observed, that if the ellipsoid abc which generates the biaxal surface of refraction be cut by a plane perpendicular to OP, the vibrations of the ray P will be parallel to the greater axis of the section, and therefore the plane of polarisation will pass through OP and the less axis; whence it is easy to show that the plane of polarisation of a ray P bisects one of the angles made by two planes intersecting in OP and passing through the nodal diameters of the surface of refraction; the bisected angle being that which contains the least semiaxis c of the generating ellipsoid. The plane of polarisation of the ray p is found in like manner. But for the rays M, m, the angle to be bisected is that which contains within it the greatest semiaxis a.

If OP' be perpendicular to a tangent plane at P, the vibrations of the ray P will be perpendicular to OP and will lie in the plane POP'. A similar remark applies to the rays M, p, m.

60. When two semiaxes a,b, of the ellipsoid abc become equal, it changes into a spheroid aac described by the revolution of the ellipse ac about the semiaxis c; and the biaxal aac, generated by this spheroid, is * composed of a sphere whose radius is a, and a concentric spheroid acc described by the revolution of the ellipse ac about the semiaxis a; so that, the diameter of the sphere being equal to the axis of revolution of the spheroid, the two surfaces touch at the extremities of the axis. This combination of a sphere and a spheroid is the surface of refraction for uniaxal crystals. In these crystals, therefore, the refracted ray whose direction is determined by the intersection of the right line RS with the surface of the sphere follows the ordinary law of a constant ratio of the sines, and is called the *ordinary* ray; whilst the other, whose variable refraction is regulated by the intersection of RS with the spheroid, is called the *extraordinary* ray. And hence uniaxal crystals are usually divided into the two classes of positive and negative, according to the character of the extraordinary ray; being called *positive* when it is the *plus* ray, and *negative* when it is the *minus* ray. The first case evidently happens when the spheroid is oblate, and therefore lies without the sphere described on its axis; the second, when the spheroid is prolate, and therefore lies within the sphere. The second case, (which is that of Iceland spar,) may be supposed to be represented in the figure (*Fig.* 8), where the elliptic section of the spheroid, made by a plane of incidence oblique to the axis, lies within the circular section of the sphere, and the minus ray is of course the extraordinary one.

61. Let PM, preserving a constant length I, move parallel to itself between the surfaces of the uniaxal sphere and spheroid, so as to form a ring (50). Then supposing the spheroid, with the ring-edge described on it by the point M, to remain fixed, imagine the sphere, carrying the ring-edge P along with it, to move parallel to PM, from P towards M, through a distance equal to I, and the two ring-edges will exactly coincide.

Hence the *uniaxal ring-edge* is the intersection of a sphere and a spheroid, the diameter of the sphere being equal to the axis of revolution of the spheroid, and the line joining their centres being perpendicular to the faces of the crystal and equal to the breadth I of the ring. And the projection of this intersection, on a plane perpendicular to the line joining the centres of the sphere and the spheroid, is the *uniaxal ring-trace*.

62. The *biaxal ring-edge* is (51) the intersection of two equal biaxal surfaces similarly posited, the line joining their centres being perpendicular to the faces of the

* Trans. R. I. A. Vol. XVI. p. 77.

crystal and equal to the breadth of the ring. And the projection of this intersection, on a plane perpendicular to the line joining the centres of the surfaces, is the *biaxal ring-trace*.*

* In applying the general theory (51, 52) to biaxal rings, it is necessary to know the equation of a biaxal surface, which may be found in the following manner. Let r, r', r'', be three rectangular radii of the generating ellipsoid abc, the two latter being the semiaxes of the section made by a plane passing through them; so that if from the centre O two distances OT, OV, equal to r', r'', be taken on the direction of r, the points T and V will belong (9) to the biaxal surface; and let a plane parallel to the plane of r', r'', and touching the ellipsoid, cut the direction of r at the distance p from the centre. Then if r make the angles α, β, γ, with the semiaxes a, b, c, we shall have, by the nature of the ellipsoid.

$$\frac{1}{r^2}=\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2},$$

$$p^2=a^2\cos^2\alpha+b^2\cos^2 b+c^2\cos^2\gamma.$$

Now since the sum of the squares of the reciprocals of three rectangular radii of an ellipsoid is constant, as well as the parallelopiped described on three conjugate semidiameters, we have the equations

$$\frac{1}{r^2}+\frac{1}{r'^2}+\frac{1}{r''^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2},$$

$$p^2r'^2r''^2=a^2b^2c^2;$$

Or,

$$\frac{1}{r'^2}+\frac{1}{r''^2}=\frac{1}{a^2}+\frac{1}{b^2}+\frac{1}{c^2}-\left(\frac{\cos^2\alpha}{a^2}+\frac{\cos^2\beta}{b^2}+\frac{\cos^2\gamma}{c^2}\right)=M,$$

$$\frac{1}{r'^2r''^2}=\frac{a^2\cos^2\alpha+b^2\cos^2\beta+c^2\cos^2\gamma}{a^2b^2c^2}=N.$$

Whence it appears that r', r'', are the values of ρ in the equation

$$\frac{1}{\rho^4}-\frac{M}{\rho^2}+N=0,$$

in which ρ denotes indifferently either semidiameter, OT or OV, of the biaxal surface. Therefore putting for M and N their values, and writing $\frac{x}{\rho}$, $\frac{y}{\rho}$, $\frac{z}{\rho}$, instead of $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and $x^2+y^2+z^2$ instead of ρ^2, we obtain, for the equation of the biaxal surface,

$$(x^2+y^2+z^2)(a^2x^2+b^2y^2+c^2z^2)-a^2(b^2+c^2)x^2-b^2(a^2+c^2)y^2-c^2(a^2+b^2)z^2+a^2b^2c^2=0.$$

This is the equation of the surface of refraction for a biaxal crystal in which a, b, c, are (54) the three principal indices of refraction, taking OS the radius of the sphere to be unity. The left-hand member of the equation is therefore the expression supplied by the theory of FRESNEL for the function U in art. 51.

When the faces of the crystal are parallel to any of the principal planes of the ellipsoid,—to the plane of xy for example,—the nature of the ring-trace may be found very easily. For if the difference of the two values of z, deduced from the preceding equation of the surface of refraction, be put equal to a constant quantity I, the result, when cleared of radicals, will be an equation of the fourth degree in x and y, which will be the equation of the corresponding ring-trace. This is a case that occurs frequently in practice; the crystal being often cut with its faces perpendicular to the axis of x or of z, because these lines bisect the angles made by the optic axes.